

Generalised continuation by means of right limits

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Abstract

Several theories have been proposed to generalise the concept of analytic continuation to holomorphic functions of the disc for which the circle is a natural boundary. Elaborating on Breuer-Simon's work on *right limits* of power series, Baladi-Marmi-Sauzin recently introduced the notion of *renascent right limit* and *rrl-continuation*.

We discuss a few examples and consider particularly the classical example of *Poincaré simple pole series* in this light. These functions are represented in the disc as series of infinitely many simple poles located on the circle; they appear for instance in small divisor problems in dynamics. We prove that any such function admits a unique rrl-continuation, which coincides with the function obtained outside the disc by summing the simple pole expansion.

1 Introduction

When one is given a function g holomorphic in the unit disc \mathbb{D} , one can ask whether g is related in some way to a holomorphic function defined outside the disc. A first answer to the question comes from Weierstraß's notion of analytic continuation. Given a point λ on the unit circle, if there exists a neighbourhood V and a holomorphic function on V whose restriction to $V \cap \mathbb{D}$ is g , then we say that λ is a *regular point* and the restriction of g to the outer part of V is an analytic continuation. If there is no regular point on the unit circle, then we say that the unit circle is a *natural boundary* for g , but is it the end of the story?

It is the purpose of “generalised analytic continuation” to investigate this situation and suggest other ways in which an outer function can be related to the inner function g . The reader is referred to the monograph [RS02] for a panorama of various theories which have been proposed to go beyond Weierstraß's point of view on analytic continuation.

This paper deals with a particular case of generalised analytic continuation, called *rrl-continuation*, which was put forward in the recent article [BMS12]. This notion is based on the *right limits* introduced in [BS11] as a tool unifying various classical criteria to detect a natural boundary. In the first part of the article, we review the definition of rrl-continuation, illustrate

it on examples and discuss ways in which it can be useful in the theory of dynamical systems.

In the second part, we apply the theory to the classical situation, first considered by Poincaré in 1883, where $g(z)$ is defined for $|z| < 1$ as a series of simple poles $\sum \frac{\rho_n}{z - e^{i\theta_n}}$, where the points $e^{i\theta_n}$ are dense on the unit circle and the nonzero complex numbers ρ_n form an absolutely convergent series. For such a function the unit circle is a natural boundary in the classical sense; however, there is a natural candidate for the outer function, namely the sum $h(z)$ of the simple pole series for $|z| > 1$. We shall prove (Theorem 3.4) that every simple pole series $g(z)$ inside the disc has a unique rrl-continuation, which coincides with the outer function $h(z)$.

2 Continuation by renascent right limits and dynamical examples

2.1 Preliminaries

We are interested in holomorphic functions defined in the unit disc by power series of the form

$$g(z) = \sum_{k=0}^{\infty} a_k z^k \quad (2.1)$$

with a bounded sequence of coefficients $\{a_k\}_{k \in \mathbb{N}}$. Our aim is to investigate the possibility of defining ‘generalised analytic continuations’ for $|z| > 1$ even when the unit circle is a natural boundary. We shall even accept what is called a strong natural boundary in [BS11]:

Definition 2.1 ([BS11]). A function $g(z)$ holomorphic in the unit disc is said to have a *strong natural boundary on the unit circle* if, for every nonempty interval (ω_1, ω_2) ,

$$\sup_{0 < r < 1} \int_{\omega_1}^{\omega_2} |g(re^{i\omega})| d\omega = \infty. \quad (2.2)$$

Clearly, if the unit circle is a strong natural boundary for g , then the unit circle is a natural boundary in the usual sense, since the function is not even bounded in any sector $\{re^{i\omega} \mid r \in (0, 1), \omega \in (\omega_1, \omega_2)\}$ (as would be the case if there were a regular point on the unit circle).

The article [BS11] provides a remarkable criterium to detect strong natural boundaries (Theorem 2.5 below), based on the notion of right limit that we now recall.

Definition 2.2 ([BS11]). (i) Let $\underline{a} = \{a_k\}_{k \in \mathbb{N}}$ be a sequence in a topological space E . A *right limit* of \underline{a} is any two-sided sequence $\underline{b} = \{b_n\}_{n \in \mathbb{Z}}$ of E for which there exists an increasing sequence of positive integers

$\{k_j\}_{j \geq 1}$ such that

$$\lim_{j \rightarrow \infty} a_{n+k_j} = b_n \quad \text{for every } n \in \mathbb{Z}. \quad (2.3)$$

- (ii) Let g be a holomorphic function of the unit disc. We say that \underline{b} is a *right limit of g* if the sequence \underline{a} formed by the Taylor coefficients at the origin, $a_k := g^{(k)}(0)/k!$, is bounded and \underline{b} is a right limit of \underline{a} .

In view of (2.3), each b_n must be an accumulation point of \underline{a} . When E is a compact metric space, every sequence \underline{a} admits right limits; given $\ell \in \mathbb{Z}$ and c accumulation point of \underline{a} , one can always find a right limit \underline{b} such that $b_\ell = c$ (see *e.g.* [BMS12, Lemma 2.1]).

In the case of a function g with bounded Taylor coefficients, each right limit gives rise to two generating series which will play an important role when investigating the boundary behaviour of g :

Definition 2.3. Given a two-sided bounded sequence of complex numbers $\underline{b} = \{b_n\}_{n \in \mathbb{Z}}$, we define the *inner and outer functions associated with \underline{b}* as

$$\begin{aligned} g_{\underline{b}}^+(z) &= \sum_{n \geq 0} b_n z^n, & z \in \mathbb{D}, \\ g_{\underline{b}}^-(z) &= - \sum_{n < 0} b_n z^n, & z \in \mathbb{E}, \end{aligned}$$

where $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ is the unit disc and $\mathbb{E} = \{z \in \mathbb{C} \mid |z| > 1\} \cup \{\infty\}$ is a disc centred at ∞ in the Riemann sphere $\widehat{\mathbb{C}}$.

Definition 2.4 ([BS11]). Given an arc J of the unit circle, $\underline{b} = \{b_n\}_{n \in \mathbb{Z}}$ is said to be *reflectionless on J* if $g_{\underline{b}}^+$ has an analytical continuation in a neighbourhood U of J in \mathbb{C} and this analytical continuation coincides with $g_{\underline{b}}^-$ on $U \cap \mathbb{E}$.

(This terminology, introduced in [BS11], comes from the spectral theory of Jacobi matrices, it is not related to the Schwarz reflection principle.)

Theorem 2.5 (Breuer–Simon, [BS11]). *Let g be holomorphic in \mathbb{D} with bounded Taylor coefficients at 0.*

- (i) *Consider a nonempty interval (ω_1, ω_2) and the corresponding arc of the unit circle $J = \{e^{i\omega} \mid \omega \in (\omega_1, \omega_2)\}$, and assume that (2.2) is violated. Then every right limit of g is reflectionless on J .*
- (ii) *If $\underline{b} = \{b_n\}_{n \in \mathbb{Z}}$ and $\tilde{\underline{b}} = \{\tilde{b}_n\}_{n \in \mathbb{Z}}$ are two distinct right limits of g and if there exists $N \in \mathbb{Z}$ such that either $b_n = \tilde{b}_n$ for all $n \geq N$ or $b_n = \tilde{b}_n$ for all $n \leq N$, then the unit circle is a strong natural boundary for g .*

2.2 The rrl-continuable functions

Breuer–Simon’s work motivated the following

- Definition 2.6** ([BMS12]). (i) A *renascent right limit* of a sequence \underline{a} in a topological space is any right limit $\underline{b} = \{b_n\}_{n \in \mathbb{Z}}$ of \underline{a} such that $b_n = a_n$ for all $n \geq 0$.
- (ii) An *rrl-continuable function* is a holomorphic function g which admits a renascent right limit \underline{b} ; then $g_{\underline{b}}^+ = g$ in \mathbb{D} and the function $g_{\underline{b}}^-$, which is holomorphic in \mathbb{E} and vanishes at ∞ , is said to be an *rrl-continuation of g* .
- (iii) An rrl-continuable function g is said to be *uniquely rrl-continuable* if it has a unique rrl-continuation; in the opposite case, it is said to be *polygenous*.

Here is the motivation behind these definitions:

Proposition 2.7. *Let g be an rrl-continuable function. Then*

- (i) *either there is an arc of the unit circle through which g admits analytic continuation; then g is uniquely rrl-continuable and all the analytic continuations of g through arcs of the unit circle match and coincide with the rrl-continuation of g ,*
- (ii) *or the unit circle is a strong natural boundary for g .*

If g is polygenous, then the unit circle is a strong natural boundary for g .

Proof. Suppose that \underline{b} is a renascent right-limit of g . If there exists an arc J in the neighbourhood of which $g = g_{\underline{b}}^+$ admits an analytic continuation, then Theorem 2.5(i) implies that this continuation coincides with the rrl-continuation $g_{\underline{b}}^-$ in \mathbb{E} . If on the contrary there is no analytic continuation for g across any arc of the unit circle, then the given renascent right limit is not reflectionless on any arc and the unit circle must be a strong natural boundary.

The last statement follows from Theorem 2.5(ii). \square

Example 2.8. In the case of a preperiodic sequence, $g(z) = \sum_{k=0}^{\infty} a_k z^k$ with $a_k = a_{k+p}$ for all $k \geq m$, one checks easily that there is no renascent right limit unless $m = 0$, *i.e.* the sequence is periodic, in which case $g(z) = (a_0 + a_1 z + \cdots + a_{p-1} z^{p-1}) / (1 - z^p)$ is rational and uniquely rrl-continuable.

More generally, *any rational function which is regular on the Riemann sphere minus the unit circle and whose poles are simple is uniquely rrl-continuable*; this follows from Theorem 3.4 below (we shall see that one can

even afford for an infinite set of poles on the unit circle—of course the function is then no longer rational). Notice that we restrict ourselves to simple poles because we consider only the case of bounded Taylor coefficients.

We emphasize that a holomorphic function g with bounded Taylor coefficients may have no rrl-continuation at all, independently of whether the unit circle is a natural boundary or not. For instance, if the sequence of Taylor coefficients of g at the origin tends to 0, then the only right limit of g is $b_n \equiv 0$ and g cannot be rrl-continuable unless $g(z) \equiv 0$; the previous example also shows that no nonzero polynomial is rrl-continuable.

Observe also that if two holomorphic functions of \mathbb{D} differ by a function h which is holomorphic in a disc $\{|z| < R\}$ with $R > 1$, then they have the same right limits; for instance, for any such h , the function $h(z) + (1 - z)^{-1}$ has only one right limit, the constant sequence $b_n \equiv 1$, but only when $h(z) \equiv 0$ is this right limit a renascent one.

When an rrl-continuable function g has a natural boundary on the unit circle, we may still think of the rrl-continuations of g as being somewhat “connected” to g and consider them as a kind of generalised analytic continuation, and the case of a unique rrl-continuation may then be particularly interesting.

Notice that even with the usual analytic continuation it may happen that, for a given $g \in \mathcal{O}(\mathbb{D})$, there are several arcs through which analytic continuation is possible but leads to different results. Think *e.g.* of $(1 - z)^{1/2}(1 + z)^{1/2}$. However, in view of Proposition 2.7, such examples are not rrl-continuable.

2.3 Power series related to dynamical systems

A first interesting class of power series which arises in connection with dynamical systems is as follows:

Definition 2.9. Let E be a metric space and $T: E \rightarrow E$ be a continuous map. Given $\gamma \in E$, we consider its orbit $\{\gamma_k = T^k(\gamma)\}_{k \geq 0}$. Then, for any bounded function $\varphi: E \rightarrow \mathbb{C}$, we say that the sequence

$$a_k = \varphi(T^k(\gamma)), \quad k \geq 0$$

is generated by the dynamical system T (in that situation φ is called an *observable*).

To determine the right limits of the power series $\sum_{k=0}^{\infty} \varphi(T^k(\gamma))z^k$, one may try to determine first the right limits of the orbit $\{T^k(\gamma)\}$ itself (and then to exploit continuity or discontinuity properties of the observable φ). The following result is a generalisation of a lemma proved in [BMS12].

Lemma 2.10. *The right limits of the orbit $\{\varphi(T^k(\gamma))\}_{k \geq 0}$ are exactly the full orbits¹ of T which are contained in the ω -limit set $\omega(\gamma, T)$.*

¹*i.e.* the two-sided sequences $\{x_n\}_{n \in \mathbb{Z}}$ of E such that $x_{n+1} = T(x_n)$ for all $n \in \mathbb{Z}$

Proof. Adapt the argument given in [BMS12, Lemma 2.4] for the case where $\{\gamma_k\}$ is dense in E . \square

One finds in [BMS12, Theorem 2] an example of such a sequence $a_k = \varphi(T^k(\gamma))$ which has uncountably many reascent right limits: the corresponding generating series is highly polygenous (in that example the observable φ is continuous but the dynamics T is a non-invertible map of a compact interval of \mathbb{R} ; the non-invertibility helps construct a huge set of full orbits).

Example 2.11. The arithmetic example due to Hecke

$$g_H(z) = \sum_{k=1}^{\infty} \{k\theta\} z^k, \quad z \in \mathbb{D}, \quad (2.4)$$

where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $\{\cdot\}$ denotes the fractional part, was shown to have a strong natural boundary in [BS11]. This can be viewed as a series generated by the translation $x \mapsto x + \theta$ on \mathbb{R}/\mathbb{Z} for a discontinuous observable.

We shall see in Proposition 2.14 that $g_H(z)$ is uniquely rrl-continuable and that $z^{-1}g_H(z)$ has exactly two rrl-continuations. Notice that, given $r \in \mathbb{N}^*$ and $g \in \mathcal{O}(\mathbb{D})$ divisible by z^r ,

$$h \text{ is a rrl-continuation of } g \implies z^{-r}h(z) \text{ is a rrl-continuation of } z^{-r}g(z),$$

but the converse is not necessarily true.

Another class of examples arises from symbolic dynamics: if $E = \bigcup_{k=1}^n P_k$ is a partition of the phase space in a finite number of sets, we can define the piecewise constant observable

$$\varphi(x) := c_k \quad \text{for } x \in P_k,$$

for some choice of constants c_1, \dots, c_n . Then, given a point x , the corresponding sequence generated by a dynamical system $T: E \rightarrow E$ is called *itinerary* of x :

$$\mathbf{itin}(x) := \{\varphi(T^k(x))\}_{k \geq 0}.$$

A powerful application is Milnor–Thurston’s *kneading theory* [MT88]. Let $T: [0, 1] \rightarrow [0, 1]$ be a continuous, unimodal map, with $T(0) = T(1) = 0$ and a critical point $c \in (0, 1)$ which we assume non-periodic for simplicity; we consider the piecewise constant observable φ which takes the value 1 on $[0, c]$ and -1 on $(c, 1]$. The *kneading sequence* $\{\epsilon_k\}_{k \geq 0}$ of T is defined to be the itinerary of c . The *kneading determinant* is the power series

$$D(z) := 1 + \sum_{k \geq 1} \epsilon_1 \cdots \epsilon_k z^k.$$

One of the applications of the kneading determinant is to find the topological entropy of the map. Indeed, if s is the smallest real zero of $D(z)$, then the entropy of T equals $-\log s$ ([MT88], Theorem 6.3).

Example 2.12. As an example, consider $T(z) := z^2 + c$ with c the Feigenbaum parameter ($c \cong -1.401155189\dots$). Then its kneading determinant is known to be

$$D(z) = \sum_{n=0}^{\infty} (-1)^{\tau_n} z^n,$$

where $\tau := (01101001\dots)$ is the *Thue-Morse sequence* generated by the substitution $0 \rightarrow 01, 1 \rightarrow 10$, starting with 0. Notice that, by the defining relation of τ , it is not hard to prove that

$$D(z) = \prod_{m=0}^{\infty} (1 - z^{2^m})$$

(from which it follows that the entropy of T is 0). One can check that $D(z)$ has precisely two renascent right limits, hence the unit circle is a strong natural boundary.

A thorough investigation of the applications to symbolic dynamics will be the object of a forthcoming article.

2.4 A theorem about the series generated by a circle map

The following result is a variant of a theorem proved in [BS11] and used there to show that Hecke's example has a strong natural boundary on the unit circle. It deals with the series generated by a dynamical system on the circle, with a special kind of observable:

Theorem 2.13. *Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Suppose that $f: \mathbb{T} \rightarrow \mathbb{T}$ is a homeomorphism and that $x^* \in \mathbb{T}$ is such that $\{f^k(x^*)\}_{k \in \mathbb{N}}$ is dense in \mathbb{T} . Suppose that a bounded function $\varphi: \mathbb{T} \rightarrow \mathbb{C}$ is continuous on $\mathbb{T} \setminus \Delta$, where $\Delta \subset \mathbb{T}$ has empty interior, and that each point of Δ is a point of discontinuity for φ at which right and left limits exist and φ is either right- or left-continuous. Let*

$$g(z) := \sum_{k=0}^{\infty} \varphi(f^k(x^*)) z^k, \quad z \in \mathbb{D}.$$

Then:

- (i) *If $f^k(x^*) \notin \Delta$ for all $k > 0$, then g is rrl-continuable.*
- (ii) *If $f^k(x^*) \notin \Delta$ for all $k \geq 0$ and there exists $n < 0$ such that $f^n(x^*) \in \Delta$, then g has at least two different rrl-continuations.*

Proof. Let us use the notation

$$y_j \xrightarrow{>} y^*, \quad \text{resp.} \quad y_j \xrightarrow{<} y^*,$$

if $\{y_j\}_{j \geq 1}$ is a sequence and y^* is a point in \mathbb{T} for which there exist lifts \tilde{y}_j and \tilde{y}^* in \mathbb{R} such that $\lim_{j \rightarrow \infty} \tilde{y}_j = \tilde{y}^*$ and, for j large enough, $\tilde{y}^* < \tilde{y}_j < \tilde{y}^* + \frac{1}{2}$, resp. $\tilde{y}^* - \frac{1}{2} < \tilde{y}_j < \tilde{y}^*$. We set

$$x_n := f^n(x^*), \quad n \in \mathbb{Z}$$

and notice that, by the density of $\{x_k\}_{k \geq 0}$ in \mathbb{T} , for every $y^* \in \mathbb{T}$ one can find increasing integer sequences $\{k_j^+\}_{j \geq 1}$ and $\{k_j^-\}_{j \geq 1}$ such that $x_{k_j^\pm} \xrightarrow{\geq} y^*$.

Suppose first that $f^k(x^*) \notin \Delta$ for all $k > 0$. Let us choose an increasing integer sequence $\{k_j\}_{j \geq 1}$ such that $x_{k_j} \xrightarrow{\epsilon} x^*$ with ϵ standing for ' $>$ ', unless $x^* \in \Delta$ and φ is left-continuous at x^* , in which case ϵ stands for ' $<$ '. Then, for every $n \in \mathbb{Z}$, $x_{n+k_j} = f^n(x_{k_j}) \xrightarrow{\epsilon_n} f^n(x^*) = x_n$ with ϵ_n standing for ' $>$ ' or ' $<$ ' according as f^n preserves or reverses orientation, and $b_n = \lim_{j \rightarrow \infty} \varphi(x_{n+k_j})$ exists by right- or left-continuity of φ at x_n . Now, for $n > 0$, we have $x_n \notin \Delta$, hence $b_n = \varphi(x_n)$; for $n = 0$, we also have $b_0 = \varphi(x_0)$ even if $x_0 = x^* \in \Delta$ thanks to our choice of $\{k_j\}$; therefore we have found a renascent right limit for $\{\varphi(x_k)\}_{k \in \mathbb{N}}$.

Suppose now that $f^k(x^*) \notin \Delta$ for all $k \geq 0$ and that one can pick $\ell > 0$ such that $f^{-\ell}(x^*) \in \Delta$. Let us choose increasing integer sequences $\{k_j^+\}_{j \geq 1}$ and $\{k_j^-\}_{j \geq 1}$ such that $x_{k_j^\pm} \xrightarrow{\geq} x_{-\ell}$. For each $n \in \mathbb{Z}$, we have $x_{n+\ell+k_j^\pm} \xrightarrow{\epsilon_n} x_n$, with ϵ_n depending on whether f^n preserves or reverses orientation, and $b_n^\pm = \lim_{j \rightarrow \infty} \varphi(x_{n+\ell+k_j^\pm})$ exists by right- or left-continuity of φ at x_n . For $n \geq 0$, φ is continuous at x_n , thus $b_n = \varphi(x_n)$, but for $n = -\ell$ we have

$$b_{-\ell}^+ = \lim_{x \xrightarrow{>} x_{-\ell}} \varphi(x) \neq b_{-\ell}^- = \lim_{x \xrightarrow{<} x_{-\ell}} \varphi(x),$$

which means that we have two different renascent right limits. \square

We now discuss the rrl-continuity of Hecke's example.

Proposition 2.14. *Let us fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and denote the fractional part function by $\{\cdot\}$. Then the function $g_H(z) = \sum_{k=1}^{\infty} \{k\theta\}z^k$ has a unique rrl-continuation, which is*

$$g_H^-(z) = - \sum_{n < 0} \{n\theta\}z^n = g_H(z^{-1}) + (1-z)^{-1}, \quad z \in \mathbb{E}. \quad (2.5)$$

Moreover, the function $z^{-1}g_H(z) = \sum_{k=0}^{\infty} \{(k+1)\theta\}z^k$ has exactly two rrl-continuations:

$$-\sum_{n<0} \{(n+1)\theta\}z^n = z^{-1}g_H^-(z) \quad \text{and} \quad -z^{-1} + z^{-1}g_H^-(z), \quad (2.6)$$

and the unit circle is a strong natural boundary both for $g_H(z)$ and $z^{-1}g_H(z)$.

Proof. Hecke's example falls into Case (i) of Theorem 2.13: denoting by $\pi: \mathbb{R} \rightarrow \mathbb{T}$ the canonical projection, we can define the homeomorphism f by $f \circ \pi(\tilde{x}) = \pi(\tilde{x} + \theta)$ and the discontinuous observable φ^+ by $\varphi^+ \circ \pi(\tilde{x}) = \tilde{x} - \lfloor \tilde{x} \rfloor$, then $g_H(z) = \sum_{k=0}^{\infty} a_k z^k$ with $a_k := \varphi^+(f^k(x^*))$ and $x^* := \pi(0)$, and Δ is reduced to $\{x^*\}$ in that case, with φ^+ right-continuous.

Therefore, g_H has at least one rrl-continuation. To show its uniqueness, we observe that the canonical projection π is a continuous right inverse for $\varphi^+: \mathbb{T} \rightarrow \mathbb{R}$; for every right limit $\underline{b} = \{b_n\}_{n \in \mathbb{Z}}$ of $\{a_k\}_{k \in \mathbb{N}}$, this implies that $y_n := \pi(b_n)$ defines a right limit for $\{f^k(x^*)\}_{k \in \mathbb{N}}$, and the continuity of f and f^{-1} clearly implies $y_n = f^n(y_0)$; if \underline{b} is a renascent right limit, then $b_0 = 0$, hence $y_0 = x^*$ and $y_n \notin \Delta$ for $n \in \mathbb{Z}^*$, and $b_n = \varphi^+(y_n) = \varphi^+(f^n(x^*))$ by continuity of φ^+ on $\mathbb{T} \setminus \Delta$. We thus obtain (2.5) (the representation of g_H^- as $g_H(z^{-1}) + (1-z)^{-1}$ stems from $\tilde{x} \in \mathbb{R} \setminus \mathbb{Z} \implies -\{-\tilde{x}\} = \{\tilde{x}\} - 1$).

On the other hand, the function $g(z) := z^{-1}g_H(z)$ falls into Case (ii) of Theorem 2.13. Indeed, the only difference with the previous case is the initial condition, $x_1^* = \pi(\theta)$. Therefore, g has at least two rrl-continuations. By Theorem 2.5(ii), it follows that the unit circle is a strong natural boundary for g , and thus also for g_H .

Arguing as above, we see that any renascent right limit of g is of the form $\underline{b} = \{b_n\}_{n \in \mathbb{Z}}$ with $\pi(b_n) = f^n(x_1^*)$; now $\pi(b_n) \in \mathbb{T} \setminus \Delta$ only for $n \in \mathbb{Z} \setminus \{-1\}$, while $\pi(b_{-1}) = \pi(0)$. Since $b_{-1} \in [0, 1]$, we conclude that there are only two possibilities: $b_{-1} = 0$ or 1 . Both cases are possible (if not there would be only one renascent right limit), we thus find the two rrl-continuations indicated in (2.6).

Notice that these functions can be written $-\sum_{n<0} \varphi^\pm(f^n(x_1^*))z^n$, with a function $\varphi^-: \mathbb{T} \rightarrow \mathbb{R}$ defined by $\varphi^- \circ \pi(\tilde{x}) = \tilde{x} - \lceil \tilde{x} \rceil$ which is left-continuous at $\pi(0) = f^{-1}(x_1^*)$. \square

Corollary 2.15. *With the same assumptions and notations as in Proposition 2.14, defining*

$$g_{H,\gamma}(z) := \sum_{k=0}^{\infty} \{\gamma + k\theta\}z^k, \quad z \in \mathbb{D}$$

for every $\gamma \in \mathbb{R} \setminus (\mathbb{Z} + \theta\mathbb{Z})$, one gets a unique rrl-continuation for $g_{H,\gamma}$:

$$g_{H,\gamma}^-(z) = -\sum_{n<0} \{\gamma + n\theta\}z^n = g_{H,-\gamma}(z^{-1}) + z(1-z)^{-1} + \{\gamma\}, \quad (2.7)$$

and the unit circle is a strong natural boundary.

Proof. The existence of the rrl-continuation is guaranteed by Theorem 2.13(i), exactly as in the proof of Proposition 2.14. One finds that (2.7) is the only possible rrl-continuation by following the same lines.

Since 0 is an accumulation point of the sequence $\{a_k = \{\gamma + k\theta\}\}_{k \in \mathbb{N}}$, we can find a right limit $\{b_n\}_{n \in \mathbb{Z}}$ such that $b_0 = 0$. With the same notations for π , f and φ^+ as in the proof of Proposition 2.14, since π is continuous, it maps $\{b_n\}$ onto a right limit $\{y_n\}$ of the sequence $\{\pi(a_k) = f^k(\pi(\gamma))\}$, which is necessarily of the form $y_n = f^n(y_0)$ (because f is continuous). Since $y_0 = \pi(0)$, the observable φ^+ is continuous at each of the points y_n with $n \in \mathbb{Z}^*$, hence $\{\varphi^+(y_n)\}$ is a right limit of the sequence $\{\varphi^+ \circ \pi(a_k) = a_k\}$.

We just obtained that $\{n\theta\}_{n \in \mathbb{Z}}$ is a right limit of $g_{H,\gamma}$. By virtue of Proposition 2.14, this right limit is not reflectionless on any arc, Theorem 2.5(i) thus implies that the unit circle is a strong natural boundary. \square

Remark 2.16. There is a relationship between the arithmetical properties of θ and the functions $g_{H,\gamma}$: for any $0 \leq \gamma_1 < \gamma_2 < 1$,

$$g_{H,\gamma_1}(z) - g_{H,\gamma_2}(z) - \frac{\gamma_2 - \gamma_1}{1 - z} = \sum_{k \in \mathcal{N}(\gamma_1, \gamma_2)} z^k,$$

where $\mathcal{N}(\gamma_1, \gamma_2)$ is the set of the occurrence times of the sequence $\{\{k\theta\}\}_{k \geq 0}$ in $[1 - \gamma_2, 1 - \gamma_1)$, that is $\mathcal{N}(\gamma_1, \gamma_2) := \{k \in \mathbb{N} \mid k\theta \in [-\gamma_2, -\gamma_1) + \mathbb{Z}\}$.

3 Poincaré simple pole series and rrl-continuability

3.1 Poincaré simple pole series

The second half of this article is dedicated to what is probably the simplest non-trivial situation in which one might wish to test the notion of rrl-continuability. The main theorem is Theorem 3.4 below, which was announced without a proof in [BMS12, Appendix A.2].

We shall use the same notations as in Definition 2.3 for $\widehat{\mathbb{C}}$, \mathbb{D} and \mathbb{E} . We shall denote by \mathbb{S} the unit circle, viewed as a subset of $\mathbb{C} \subset \widehat{\mathbb{C}}$.

Definition 3.1. Let $\ell^1(\mathbb{S}, \mathbb{C})$ be the set of all functions $\rho: \mathbb{S} \rightarrow \mathbb{C}$ such that the family $\{\rho(\lambda)\}_{\lambda \in \mathbb{S}}$ is summable. Given $\rho \in \ell^1(\mathbb{S}, \mathbb{C})$, its *support* is the set (finite or countably infinite)

$$\text{supp } \rho := \{\lambda \in \mathbb{S} \mid \rho(\lambda) \neq 0\} = \{\lambda_1, \lambda_2, \dots\}$$

and $\sum_{\lambda \in \mathbb{S}} |\rho(\lambda)| = \sum_m |\rho(\lambda_m)| < \infty$. We then define the *Poincaré simple pole series* (PSP-series for short) associated with ρ as the function

$$\Sigma(\rho)(z) := \sum_{\lambda \in \mathbb{S}} \frac{\rho(\lambda)}{z - \lambda}, \quad z \in \widehat{\mathbb{C}} \setminus \overline{\text{supp } \rho}, \quad (3.1)$$

and the *inner* and *outer* PSP-series by

$$\Sigma(\rho)^+ := \Sigma(\rho)|_{\mathbb{D}} \in \mathcal{O}(\mathbb{D}), \quad \Sigma(\rho)^- := \Sigma(\rho)|_{\mathbb{E}} \in \mathcal{O}(\mathbb{E}).$$

We say that the outer PSP-series $\Sigma(\rho)^-$ is *associated* with the inner PSP-series $\Sigma(\rho)^+$.

Since the series (3.1) converges normally on every compact subset of $\widehat{\mathbb{C}} \setminus \overline{\text{supp } \rho}$, the functions $\Sigma(\rho)^\pm$ are holomorphic respectively on \mathbb{D} and \mathbb{E} .

Our terminology is motivated by Poincaré's article [Po83] (see also [Po92]), where he studies this kind of series; assuming that the support of ρ is dense in \mathbb{S} , Poincaré proves that the unit circle is a natural boundary for $\Sigma(\rho)^\pm$ and he discusses the relationship between the two functions. Later Borel, Wolff and Denjoy studied such series, considering also more general distributions of poles λ (not restricted to lie on \mathbb{S}).

The subclass of PSP-series $\Sigma(\rho)$ with $\text{supp } \rho$ contained in the set of roots of unity was studied in [MS03] for dynamical reasons, in relation with small divisor problems.

Notice that one might be tempted to enlarge considerably the framework by considering series of rational functions which are regular in $\widehat{\mathbb{C}} \setminus \mathbb{S}$ rather than series of simple poles. However one must be aware that such an expansion may represent a constant function in \mathbb{D} and another constant function in \mathbb{E} , a phenomenon which certainly does not fit with our intuition of what generalized analytic continuation should be (see [Po92], [RS02]).

The class of inner PSP-series is an interesting class of functions for which we shall prove unique rrl-continuability. It clearly contains the rational functions which are regular on $\widehat{\mathbb{C}} \setminus \mathbb{S}$ and whose poles are simple, but we are more interested in the case where \mathbb{S} is a natural boundary.

Any inner PSP-series uniquely determines the associated outer PSP-series, because ρ and thus $\Sigma(\rho)^-$ are uniquely determined by $\Sigma(\rho)^+$:

Lemma 3.2. *Suppose that $\rho \in \ell^1(\mathbb{S}, \mathbb{C})$. Then*

$$\rho(\lambda) = \lim_{z \rightarrow \lambda \text{ radially}} (z - \lambda) \Sigma(\rho)^+(z) \quad \text{as } z \rightarrow \lambda \text{ radially}$$

for every $\lambda \in \mathbb{S}$. Hence the map $\rho \in \ell^1(\mathbb{S}, \mathbb{C}) \mapsto \Sigma(\rho)^+ \in \mathcal{O}(\mathbb{D})$ is injective.

Proof. Given $\lambda^* \in \mathbb{S}$, we can write

$$(z - \lambda^*) \Sigma(\rho)^+(z) - \rho(\lambda^*) = \sum_{\lambda \neq \lambda^*} \rho(\lambda) \frac{z - \lambda^*}{z - \lambda}, \quad z \in \mathbb{D}.$$

For each $\lambda \neq \lambda^*$, we have $z \in [0, \lambda^*] \Rightarrow |\frac{z - \lambda^*}{z - \lambda}| < 1$ and $\frac{z - \lambda^*}{z - \lambda} \rightarrow 0$ as $z \rightarrow \lambda^*$, whence the result follows by dominated convergence. \square

Remark 3.3. In fact, one even has

$$\rho(\lambda) = \lim_{z \xrightarrow{NT} \lambda} (z - \lambda) \Sigma(\rho)^+(z), \quad \lambda \in \mathbb{S},$$

where ‘ $\lim_{z \xrightarrow{NT} \lambda}$ ’ denotes nontangential limit.

Lemma 3.2 shows that, if $\lambda \in \text{supp } \rho$, the function $\Sigma(\rho)^+$ is not bounded on the ray $[0, \lambda]$, hence λ is necessarily a singular point of the function. This entails a dichotomy:

- (a) either $\text{supp } \rho$ is not dense in \mathbb{S} ; then $\widehat{\mathbb{C}} \setminus \overline{\text{supp } \rho}$ is connected and $\mathbb{S} \setminus \overline{\text{supp } \rho}$ is a countable union of open arcs of the unit circle, in the neighbourhood of which $\Sigma(\rho)$ is holomorphic; we can thus view $\Sigma(\rho)^+$ and $\Sigma(\rho)^-$ as the analytic continuation of each other through any of these arcs;
- (b) or $\text{supp } \rho$ is dense in the unit circle, \mathbb{D} and \mathbb{E} are the two connected components of $\widehat{\mathbb{C}} \setminus \overline{\text{supp } \rho} = \widehat{\mathbb{C}} \setminus \mathbb{S}$ and the unit circle is a natural boundary for both $\Sigma(\rho)^+$ and $\Sigma(\rho)^-$.

In the latter case, one may still study the convergence of (3.1) on the unit circle (by restricting oneself to $z \in \mathbb{S}$ “sufficiently far” from $\text{supp } \rho$) and try to “connect” the functions $\Sigma(\rho)^+$ and $\Sigma(\rho)^-$ by a property of Borel-monogenic regularity. We shall return to these questions later (Proposition 3.7).

Our main theorem about inner PSP-series is in term of the notion of generalized analytic continuation discussed in Section 2: the outer PSP-series is the unique rrl-continuation of the inner PSP-series with which it is associated.

Theorem 3.4. *Let $g \in \mathcal{O}(\mathbb{D})$ be an inner PSP-series. Then g is uniquely rrl-continuable and its rrl-continuation is the associated outer PSP-series.*

The proof will be given in Sections 4.1 and 4.2. Together with Proposition 2.7, this immediately yields

Corollary 3.5. *Assume that the unit circle is a natural boundary for an inner PSP-series g (i.e. the support of the corresponding $\rho \in \ell^1(\mathbb{S}, \mathbb{C})$ is dense in \mathbb{S}). Then the unit circle is a strong natural boundary for g .*

3.2 An example

Here is an example which is reminiscent of Hecke’s example:

Proposition 3.6. *Let $\varphi: \mathbb{R} \rightarrow \mathbb{C}$ be continuous and 1-periodic and assume that the sequence $(\hat{\varphi}(j))_{j \in \mathbb{Z}}$ of its Fourier coefficients is absolutely convergent:*

$$\sum_{j \in \mathbb{Z}} |\hat{\varphi}(j)| < \infty. \tag{3.2}$$

Then, for any $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the sum of the convergent power series

$$g^+(z) := \sum_{n \geq 0} \varphi(n\theta) z^n, \quad z \in \mathbb{D} \quad (3.3)$$

is an inner PSP-series, with associated outer PSP-series given by

$$g^-(z) := - \sum_{n < 0} \varphi(n\theta) z^n, \quad z \in \mathbb{E}. \quad (3.4)$$

Moreover, if φ is not a trigonometric polynomial, then the unit circle is a strong natural boundary for g^+ .

Proof. Use the formula (4.2) for the Taylor coefficients at 0 of an inner PSP-series and the coefficients at ∞ of the associated outer PSP-series: consider the pairwise distinct points $\lambda_j = e^{-2\pi i j \theta}$, $j \in \mathbb{Z}$, and define $\rho: \mathbb{S} \rightarrow \mathbb{C}$ by

$$\rho(\lambda_j) = -\lambda_j \hat{\varphi}(j)$$

and $\rho(\lambda) = 0$ if λ is not an integer power of $e^{-2\pi i \theta}$; then $\rho \in \ell^1(\mathbb{S}, \mathbb{C})$ and $-\sum \rho(\lambda) \lambda^{-n-1} = \sum \hat{\varphi}(j) e^{2\pi i j n \theta} = \varphi(n\theta)$, hence $g^\pm = \Sigma(\rho)^\pm$. \square

In Hecke's example, the observable φ^+ violates condition (3.2) and is not continuous. Still, the formula (2.5) that we obtained for its rrl-continuation looks like an echo of (3.4).

3.3 Monogenic regularity

Let us return to the general case of a PSP-series $\Sigma(\rho)$. We now wish to analyze the relationship between the inner and the outer functions in terms of regularity: at least for a certain class of ρ 's, the connection between them can be reinforced. The reader is referred to [MS03] or [MS11] for the definition of the space of Borel monogenic functions $\mathcal{M}((K_j), \mathbb{C})$ associated with a monotonic non-decreasing sequence of compact subsets of the Riemann sphere $\hat{\mathbb{C}}$ and to [MS11] for the notion of \mathcal{H}^1 -quasianalyticity.

Proposition 3.7. *Let $\rho \in \ell^1(\mathbb{S}, \mathbb{C})$ with infinite support*

$$\text{supp } \rho = \{\lambda_1, \lambda_2, \dots\}$$

and assume that there exists $\sigma > 1$ such that $\sum_{m=1}^{\infty} m^\sigma |\rho(\lambda_m)| < \infty$.

Then there exists an increasing sequence (K_j) of compact subsets of $\hat{\mathbb{C}}$ such that $K := \bigcup K_j$ has its complement contained in the unit circle and of zero Haar measure, and such that the space of Borel monogenic functions associated with (K_j) ,

$$\mathcal{M}((K_j), \mathbb{C}),$$

contains $\Sigma(\rho)$ and is \mathcal{H}^1 -quasianalytic.

Proof. For each $m \geq 1$, let $\omega_m \in [0, 1]$ be such that $\lambda_m = e^{2\pi i \omega_m}$. For $j \geq 1$, define

$$A_j^{\mathbb{R}} := \{ \omega \in \mathbb{R} \mid \forall m \geq 1, \forall N \in \mathbb{Z}, |\omega - \omega_m - N| \geq \frac{1}{jm^\sigma} \}$$

and observe that the Lebesgue measure of $[0, 1] \setminus A_j^{\mathbb{R}}$ is less than $2\zeta(\sigma)/j$ (where ζ denotes the Riemann zeta function). Proceed adapting [MS11, sect. 5] (or [MS03, sect. 2.4–2.5]). \square

The \mathcal{H}^1 -quasianalyticity property means that any function g of the space $\mathcal{M}((K_j), \mathbb{C})$ is determined by its restriction to any subset of K which has positive linear Hausdorff measure; in particular it is determined by its inner restriction $g|_{\mathbb{D}}$. This yields a totally different way of recovering the outer function $g|_{\mathbb{E}}$ from the inner function. In view of Theorem 3.4, one may wonder whether it is true that $g|_{\mathbb{E}}$ is the only rrl-continuation of $g|_{\mathbb{D}}$ for any $g \in \mathcal{M}((K_j), \mathbb{C})$.

A particular case of Proposition 3.7 occurs in the case of the solution $\sum_{m=1}^{\infty} G_m \frac{w^m}{q^m - 1}$ of the cohomological equation considered in [MS03] (with the “multiplier” q playing the role of the variable z of the present article).

4 Proof of Theorem 3.4

4.1 Existence of rrl-continuation by the outer function

Let $\rho \in \ell^1(\mathbb{S}, \mathbb{C})$, $g := \Sigma(\rho)^+$, $h := \Sigma(\rho)^-$. We must prove that h is an rrl-continuation of g .

The Taylor expansion of g at the origin and the Taylor expansion of h at ∞ are easily computed by expanding the geometric series $\frac{1}{z-\lambda} = -\lambda^{-1}(1 - \lambda^{-1}z)^{-1} = z^{-1}(1 - \lambda z^{-1})^{-1}$ and permuting sums; one can write the result as

$$g(z) = \sum_{n \geq 0} b_n z^n, \quad h(z) = - \sum_{n < 0} b_n z^n, \quad (4.1)$$

$$b_n := - \sum_{\lambda \in \mathbb{S}} \rho(\lambda) \lambda^{-n-1} \quad \text{for } n \in \mathbb{Z}. \quad (4.2)$$

The problem is thus to find an unbounded integer sequence $(k_j)_{j \geq 1}$ such that

$$\lim_{j \rightarrow \infty} b_{n+k_j} = b_n \quad \text{for every } n \in \mathbb{Z} \quad (4.3)$$

(indeed, from any such unbounded sequence, one can extract an increasing sequence for which (4.3) still holds, showing that $\underline{b} = (b_n)_{n \in \mathbb{Z}}$ is a right limit of $(b_k)_{k \in \mathbb{N}}$, with $g = g_{\underline{b}}^+$ and $h = g_{\underline{b}}^-$).

We have $b_{n+k_j} = -\sum_{\lambda \in \mathbb{S}} \lambda^{-k_j} \cdot \rho(\lambda) \lambda^{-n-1}$. By the dominated convergence theorem, if the sequence $(k_j)_{j \geq 1}$ satisfies

$$\lim_{j \rightarrow \infty} \lambda^{k_j} = 1 \quad \text{for each } \lambda \in \text{supp } \rho \quad (4.4)$$

then (4.3) holds. The problem thus reduces to find an unbounded integer sequence $(k_j)_{j \geq 1}$ satisfying (4.4).

1st case

Assume that $\text{supp } \rho$ is contained in the set of all roots of unity. Then one can take $k_j := j!$, since $\lambda^{j!} = 1$ for any $j \geq \text{order of } \lambda \text{ as a root of unity}$.

2nd case

Assume that $\text{supp } \rho$ is not contained in the set of all roots of unity. We write $\text{supp } \rho = \{\lambda_1, \lambda_2, \dots\}$. For each $j \geq 1$, we set

$$V_j := \{e^{2\pi i \omega} \mid 0 \leq \omega < 1/j\} \subset \mathbb{S}$$

and we consider the “cells”

$$W_{\ell_1, \ell_2, \dots, \ell_j} := e^{2\pi i \ell_1/j} V_j \times e^{2\pi i \ell_2/j} V_j \times \dots \times e^{2\pi i \ell_j/j} V_j \subset \mathbb{S}^j$$

(where $e^{2\pi i \ell_r/j} V_j$ is short-hand for $\{e^{2\pi i \omega} \mid \ell_r/j \leq \omega < (\ell_r + 1)/j\}$) for each integer j -tuple $(\ell_1, \ell_2, \dots, \ell_j)$ with $0 \leq \ell_1, \ell_2, \dots, \ell_j \leq j-1$; these are j^j cells which cover the torus \mathbb{S}^j . Now consider the $j^j + 1$ points

$$\Lambda_{j,m} := (\lambda_1^m, \lambda_2^m, \dots, \lambda_j^m) \in \mathbb{S}^j, \quad \text{for } m = 0, 1, \dots, j^j.$$

Out of them, at least two belong to the same cell, we thus can find $m_j < m'_j$ such that Λ_{j,m_j} and Λ_{j,m'_j} belong to the same cell $W_{\ell_1, \ell_2, \dots, \ell_j}$; this means that

$$\lambda_r^{m_j}, \lambda_r^{m'_j} \in e^{2\pi i \ell_r/j} V_j$$

for all $r = 1, 2, \dots, j$. This implies

$$\lambda_r^{k_j} \in V_j, \quad \text{where } k_j := m'_j - m_j \quad (4.5)$$

for all $r \leq j$. Keeping r fixed but arbitrary, we thus get $\lim_{j \rightarrow \infty} \lambda_r^{k_j} = 1$. Therefore we have obtained (4.4) with the sequence $(k_j)_{j \geq 1}$ defined by (4.5). Now this sequence cannot be bounded because, if it were, (4.4) would imply that each element of $\text{supp } \rho$ is a root of unity. This ends the first part of the proof.

4.2 Uniqueness of rrl-continuation

4.2.1 Reduction of the problem

Let $\rho \in \ell^1(\mathbb{S}, \mathbb{C})$, $g := \Sigma(\rho)^+$, $h := \Sigma(\rho)^-$. We must prove that any rrl-continuation of g coincides with h . In fact we shall prove more: using the notations (4.1)–(4.2) for the Taylor coefficients of g and h , we shall see that, for any increasing sequence of positive integers $\{k_j\}$, the property

$$\lim_{j \rightarrow \infty} b_{n+k_j} = b_n \quad \text{for each } n \geq 0 \quad (4.6)$$

implies

$$\lim_{j \rightarrow \infty} \lambda^{-k_j} = 1 \quad \text{for each } \lambda \in \text{supp } \rho. \quad (4.7)$$

By the dominated convergence theorem, this will imply

$$\lim_{j \rightarrow \infty} b_{n+k_j} = b_n \quad \text{for each } n < 0. \quad (4.8)$$

Indeed, recall that

$$b_n = - \sum_{\lambda \in \mathbb{S}} \rho(\lambda) \lambda^{-n-1} \quad \text{for } n \in \mathbb{Z},$$

hence $b_{n+k_j} = - \sum_{\lambda \in \mathbb{S}} \rho(\lambda) \lambda^{-n-1} \cdot \lambda^{-k_j}$ will tend to b_n also for negative n .

The implication (4.6) \Rightarrow (4.7) will follow from the

Proposition 4.1. *The operator $M: \ell^1(\mathbb{S}, \mathbb{C}) \rightarrow \ell^\infty(\mathbb{N})$ defined by*

$$M(\alpha)(n) := \sum_{\lambda \in \mathbb{S}} \alpha(\lambda) \lambda^n, \quad n \in \mathbb{N}$$

for every $\alpha \in \ell^1(\mathbb{S}, \mathbb{C})$ is injective.

Proposition 4.1 implies the second part of Theorem 3.4. Suppose that (4.6) holds for a sequence $\{k_j\}$, i.e.

$$\sum_{\lambda \in \mathbb{S}} \rho(\lambda) \lambda^{-n-1} \cdot \lambda^{-k_j} \xrightarrow{j \rightarrow \infty} \sum_{\lambda \in \mathbb{S}} \rho(\lambda) \lambda^{-n-1} \quad \text{for each } n \geq 0.$$

This assumption can be rewritten as

$$\langle \phi_n, \psi_{k_j} \rangle \xrightarrow{j \rightarrow \infty} \langle \phi_n, \chi \rangle \quad \text{for each } n \geq 0, \quad (4.9)$$

where the scalar products are intended in the Hilbert space $H := \ell^2(\text{supp } \rho, \mathbb{C})$ and $\phi_m, \psi_m, \chi \in H$ are defined for any $m \in \mathbb{N}$ by

$$\begin{aligned} \phi_m(\lambda) &= |\rho(\lambda)|^{1/2} e^{i\theta(\lambda)} \lambda^{-m-1}, \\ \psi_m(\lambda) &= |\rho(\lambda)|^{1/2} \lambda^m, \\ \chi(\lambda) &= |\rho(\lambda)|^{1/2}, \end{aligned}$$

for every $\lambda \in \text{supp } \rho$, with any $\theta(\lambda) \in \mathbb{R}$ such that $\rho(\lambda) = |\rho(\lambda)|e^{i\theta(\lambda)}$. Let W denote the closure of the span of $\{\phi_m\}_{m \in \mathbb{N}}$ in H . Since $\|\psi_{k_j} - \chi\|_H \leq 2\|\chi\|_H$, we get, by dominated convergence,

$$\langle \phi, \psi_{k_j} \rangle \xrightarrow{j \rightarrow \infty} \langle \phi, \chi \rangle \quad \text{for each } \phi \in W \quad (4.10)$$

Now, suppose $\beta \in W^\perp$. This means that $\beta \in \ell^2(\text{supp } \rho, \mathbb{C})$ and

$$\sum_{\lambda \in \mathbb{S}} \beta(\lambda) \bar{\phi}_0(\lambda) \lambda^n = 0 \quad \text{for each } n \geq 0$$

(because $\phi_n(\lambda) = \phi_0(\lambda) \lambda^{-n}$). Setting $\alpha(\lambda) := \beta(\lambda) \bar{\phi}_0(\lambda)$, we get $\alpha \in \ell^1(\mathbb{S}, \mathbb{C})$ such that $M(\alpha) \equiv 0$, which implies $\alpha \equiv 0$ by Proposition 4.1, hence $\beta \equiv 0$. Thus W is the whole H .

Given $\lambda \in \mathbb{S}$, we define $\delta_\lambda \in H$ by $\delta_\lambda(\lambda') = |\rho(\lambda)|^{1/2}$ if $\lambda' = \lambda$ and $\delta_\lambda(\lambda') = 0$ if $\lambda' \neq \lambda$. Then, (4.10) with $\phi = \delta_\lambda$ reads

$$\langle \delta_\lambda, \psi_{k_j} \rangle = |\rho(\lambda)| \lambda^{-k_j} \xrightarrow{j \rightarrow \infty} \langle \delta_\lambda, \chi \rangle = |\rho(\lambda)|,$$

which is (4.7). As explained above, this implies (4.8) and the proof is complete. \square

4.2.2 Proof of Proposition 4.1

We first introduce some notation: given $F \subset \mathbb{S}$ finite, we define the polynomials

$$P_F(X) := \prod_{\mu \in F} (X - \mu), \quad Q_{\lambda, F}(X) := \prod_{\mu \in F \setminus \{\lambda\}} (X - \mu) \quad \text{for each } \lambda \in F.$$

Observe that

$$Q_{\lambda, F}(X) = P_{F \setminus \{\lambda\}}(X) = \frac{P_F(X)}{X - \lambda} \quad \text{and} \quad Q_{\lambda, F}(\lambda) = P'_F(\lambda) \quad \text{for each } \lambda \in F. \quad (4.11)$$

Lemma 4.2. *Let F be a finite subset of \mathbb{S} and $\alpha \in \ell^1(\mathbb{S}, \mathbb{C})$. Then*

$$M(\alpha) = 0 \implies \alpha(\lambda) = - \sum_{\mu \in \mathbb{S} \setminus F} \frac{Q_{\lambda, F}(\mu)}{Q_{\lambda, F}(\lambda)} \alpha(\mu) \quad \text{for each } \lambda \in F.$$

Proof. The assumption $M(\alpha) = 0$ implies that, for any $Q \in \mathbb{C}[X]$,

$$\sum_{\mu \in \mathbb{S}} Q(\mu) \alpha(\mu) = 0.$$

Choosing $Q = Q_{\lambda, F}$, since $Q_{\lambda, F}(\mu) = 0$ for $\mu \in F \setminus \{\lambda\}$, we get

$$Q_{\lambda, F}(\lambda) \alpha(\lambda) + \sum_{\mu \in \mathbb{S} \setminus F} Q_{\lambda, F}(\mu) \alpha(\mu) = 0.$$

\square

This is already sufficient to conclude when the support of ρ is contained in the set of roots of unity

$$\mathcal{R} := \bigcup_{m \geq 1} \mathcal{R}_m, \quad \mathcal{R}_m := \{ \lambda \in \mathbb{S} \mid \lambda^m = 1 \} \quad \text{for } m \in \mathbb{N}^*.$$

Lemma 4.3. *Let $\alpha \in \ell^1(\mathbb{S}, \mathbb{C})$ with $\text{supp } \alpha \subset \mathcal{R}$. Then*

$$M(\alpha) = 0 \implies \alpha = 0.$$

Proof. Suppose $\text{supp } \alpha \subset \mathcal{R}$ and $M(\alpha) = 0$ and fix $\lambda \in \mathcal{R}$. For every $m \geq 1$ such that $\lambda \in \mathcal{R}_m$, Lemma 4.2 implies

$$\alpha(\lambda) = - \sum_{\mu \in \mathbb{S} \setminus \mathcal{R}_m} \frac{Q_{\lambda, \mathcal{R}_m}(\mu)}{Q_{\lambda, \mathcal{R}_m}(\lambda)} \alpha(\mu).$$

But $P_{\mathcal{R}_m}(X) = X^m - 1$ and, by (4.11),

$$Q_{\lambda, \mathcal{R}_m}(X) = \frac{X^m - 1}{X - \lambda} = \frac{X^m - \lambda^m}{X - \lambda} = \sum_{k=0}^{m-1} \lambda^{m-1-k} X^k,$$

therefore $Q_{\lambda, \mathcal{R}_m}(\lambda) = m\lambda^{-1}$ and $|Q_{\lambda, \mathcal{R}_m}(\mu)| \leq m$ for every $\mu \in \mathbb{S}$, whence $\left| \frac{Q_{\lambda, \mathcal{R}_m}(\mu)}{Q_{\lambda, \mathcal{R}_m}(\lambda)} \right| \leq 1$ and

$$|\alpha(\lambda)| \leq \sum_{\mu \in \mathbb{S} \setminus \mathcal{R}_m} |\alpha(\mu)|.$$

By choosing $m = j!$, since the sequence $\{\mathcal{R}_{j!}\}_{j \geq 1}$ exhausts \mathcal{R} , we get a sequence of inequalities in which the right-hand side tends to 0 as j tends to ∞ , whence $\alpha(\lambda) = 0$. \square

For the general case, the idea is to use Diophantine approximation to enrich any finite subset of $\text{supp } \alpha$ so as to make it “close enough” to one of the sets \mathcal{R}_m . By “close enough”, we mean that $P_F(X)$ is close to $P_{\mathcal{R}_m}(X) = X^m - 1$ in the following sense:

Definition 4.4. Let $\varepsilon \in (0, 1)$. A finite subset F of \mathbb{S} is said to be ε -balanced if

$$\left\| P_F(X) - (X^{|F|} - 1) \right\|_1 \leq \varepsilon,$$

where $|F|$ denotes the cardinality of F and $\|\cdot\|_1: \mathbb{C}[X] \rightarrow \mathbb{R}^+$ is defined by

$$\|a_0 + a_1X + \cdots + a_dX^d\|_1 := |a_0| + |a_1| + \cdots + |a_d|.$$

Lemma 4.5. *Suppose that $F \subset \mathbb{S}$ has finite cardinality $m \geq 1$. Let $\lambda \in F$. Then*

$$(i) \text{ one has } \|Q_{\lambda, F}\|_1 \leq m \|P_F\|_1;$$

(ii) if moreover F is ε -balanced for a given $\varepsilon \in (0, 1)$, then

$$\max_{\mu \in \mathbb{S}} |Q_{\lambda, F}(\mu)| \leq m(2 + \varepsilon) \quad \text{and} \quad |Q_{\lambda, F}(\lambda)| \geq m(1 - \varepsilon).$$

Proof. We can write $P_F(X) = a_0 + a_1X + \cdots + a_mX^m$ with $a_m = 1$ and $Q_{\lambda, F}(X) = b_0 + b_1X + \cdots + b_{m-1}X^{m-1}$.

(i) The identity $P_F(X) = (X - \lambda)Q_{\lambda, F}(X)$ implies

$$\begin{aligned} a_0 &= -\lambda b_0 \\ a_1 &= -\lambda b_1 + b_0 \\ &\vdots \\ a_{m-1} &= -\lambda b_{m-1} + b_{m-2}. \end{aligned}$$

This entails $|b_0| \leq |a_0|$ and $|b_k| \leq |a_k| + |b_{k-1}|$ for $k = 1, \dots, m-1$, whence $|b_k| \leq |a_0| + \cdots + |a_k| \leq \|P_F\|_1$ for $k = 0, \dots, m-1$, and finally $\|Q_{\lambda, F}\|_1 \leq m\|P_F\|_1$.

(ii) The ε -balancedness assumption reads

$$|a_0 - 1| + |a_1| + \cdots + |a_{m-1}| \leq \varepsilon. \quad (4.12)$$

On the one hand, we have $\|P_F\|_1 \leq \|X^m - 1\|_1 + \varepsilon = 2 + \varepsilon$, hence $\max_{\mathbb{S}} |Q_{\lambda, F}| \leq \|Q_{\lambda, F}\|_1 \leq m(2 + \varepsilon)$ by (i). On the other hand, by (4.11), $Q_{\lambda, F}(\lambda) = P'_F(\lambda) = m\lambda^{m-1} + \sum_{k=1}^{m-1} k a_k \lambda^{k-1}$ and (4.12) implies that $\left| \sum_{k=1}^{m-1} k a_k \lambda^{k-1} \right| \leq m\varepsilon$, whence the conclusion follows. \square

Now, by adapting the proof of Lemma 4.3, we get

Lemma 4.6. *Let $\varepsilon \in (0, 1)$. Let $\alpha \in \ell^1(\mathbb{S}, \mathbb{C})$ with $\text{supp } \alpha \subset \bigcup_{j \geq 1} F_j$, where $\{F_j\}_{j \geq 1}$ is an increasing sequence of ε -balanced subsets of \mathbb{S} . Then*

$$M(\alpha) = 0 \implies \alpha = 0.$$

Proof. Suppose $M(\alpha) = 0$ and fix $\lambda \in \mathcal{R}$. For every $j \geq 1$ such that $\lambda \in F_j$, Lemma 4.2 implies

$$\alpha(\lambda) = - \sum_{\mu \in \mathbb{S} \setminus F_j} \frac{Q_{\lambda, F_j}(\mu)}{Q_{\lambda, F_j}(\lambda)} \alpha(\mu)$$

and Lemma 4.5(ii) then yields

$$|\alpha(\lambda)| \leq \frac{2 + \varepsilon}{1 - \varepsilon} \sum_{\mu \in \mathbb{S} \setminus F_j} |\alpha(\mu)|.$$

The right-hand side tends to 0 as j tends to ∞ , therefore $\alpha(\lambda) = 0$. \square

Now, the following Diophantine approximation result will be sufficient to conclude:

Lemma 4.7. *For any finite subset G of \mathbb{S} there exists a finite subset H of \mathbb{S} such that $G \cup H$ is $\frac{1}{2}$ -balanced.*

Proof of Lemma 4.7. Let us write $\lambda = e^{2\pi i\theta(\lambda)}$ with $\theta(\lambda) \in \mathbb{R}$ for each $\lambda \in G$. Let $m := |G|$. For each integer $M \geq 1$, Dirichlet's theorem ([Ca57], Thm VI) yields integers N and $\{p(\lambda)\}_{\lambda \in G}$ such that $1 \leq N \leq M$ and

$$\left| \theta(\lambda) - \frac{p(\lambda)}{N} \right| \leq \frac{1}{NM^{1/m}} \quad \text{for each } \lambda \in G.$$

We shall see that

$$H := \mathcal{R}_N \setminus \{e^{2\pi i p(\lambda)/N}\}_{\lambda \in G}$$

satisfies the required properties provided M is large enough.

Let $F := G \cup H$: this set is obtained from the set of all N th order roots of unity by replacing those of the form $e^{2\pi i p(\lambda)/N}$ by the corresponding $e^{2\pi i\theta(\lambda)}$. Choosing $M > 2^m$, we ensure that $|F| = N$ (because for $p \neq p'$, $\left| \frac{p'}{N} - \frac{p}{N} \right| \geq \frac{1}{N}$ cannot be $\leq \frac{2}{NM^{1/m}}$) and there is a bijection $\xi: \mathcal{R}_N \rightarrow F$ such that

$$\begin{aligned} \mu \in \xi^{-1}(H) &\implies \xi(\mu) = \mu, \\ \mu \in \xi^{-1}(G) &\implies \xi(\mu) = \lambda \text{ such that } \mu = e^{2\pi i p(\lambda)/N}. \end{aligned}$$

We can set $\delta(\mu) := \mu - \xi(\mu)$, so that

$$\begin{aligned} \mu \in \xi^{-1}(H) &\implies \delta(\mu) = 0, \\ \mu \in \xi^{-1}(G) &\implies |\delta(\mu)| \leq \frac{2\pi}{NM^{1/m}}. \end{aligned}$$

We want $\|P_F - P_{\mathcal{R}_N}\|_1 \leq \frac{1}{2}$. We have

$$P_F(X) = \prod_{\mu \in \mathcal{R}_N} (X - \xi(\mu)) = \prod_{\mu \in \mathcal{R}_N} (\delta(\mu) + X - \mu) = \sum_{K \subset \mathcal{R}_N} \delta_K P_{\mathcal{R}_N \setminus K}(X)$$

with the notation $\delta_K := \prod_{\mu \in K} \delta(\mu)$ for any subset K of \mathcal{R}_N . Of course $\delta_\emptyset = 1$ and $K \not\subset \xi^{-1}(G) \Rightarrow \delta_K = 0$, hence

$$P_F(X) - P_{\mathcal{R}_N}(X) = \sum_{\emptyset \neq K \subset \xi^{-1}(G)} \delta_K P_{\mathcal{R}_N \setminus K}(X). \quad (4.13)$$

Let $K \subset \xi^{-1}(G)$ with $|K| = k \geq 1$. On the one hand, $|\delta_K| \leq \left(\frac{2\pi}{NM^{1/m}}\right)^k$. On the other hand, Lemma 4.5(i) says that $\|P_{L \setminus \{\mu\}}\|_1 \leq |L| \|P_L\|_1$; by induction on k , this yields $\|P_{\mathcal{R}_N \setminus K}\|_1 \leq N(N-1) \cdots (N-k+1) \|P_{\mathcal{R}_N}\|_1 \leq 2N^k$,

whence $\|\delta_K P_{\mathcal{R}_N \setminus K}\|_1 \leq 2\left(\frac{2\pi}{M^{1/m}}\right)^k \leq 2\frac{(2\pi)^m}{M^{1/m}}$. Since there are at most 2^m choices for K as a subset of $\xi^{-1}(G)$, (4.13) then yields $\|P_F - P_{\mathcal{R}_N}\|_1 \leq 2^{m+1}\frac{(2\pi)^m}{M^{1/m}}$, which is $\leq \frac{1}{2}$ for M large enough. \square

End of the proof of Proposition 4.1. Suppose $\alpha \in \ell^1(\mathbb{S}, \mathbb{C})$ and $M(\alpha) = 0$. Since $\text{supp } \alpha$ is countable, we can find a sequence $\{F_j^*\}_{j \geq 1}$ of finite subsets of \mathbb{S} such that $\text{supp } \alpha \subset \bigcup_{j \geq 1} F_j^*$. We proceed by induction to construct an increasing sequence $\{F_j\}_{j \geq 1}$ of $\frac{1}{2}$ -balanced subsets of \mathbb{S} such that

$$F_k^* \subset F_k \quad \text{for each } k \in \mathbb{N}^*. \quad (4.14)$$

For $j = 1$, Lemma 4.7 yields a finite $H_1 \subset \mathbb{S}$ such that $F_1 := F_1^* \cup H_1$ is $\frac{1}{2}$ -balanced. For $j > 1$, assuming to have already defined $\frac{1}{2}$ -balanced subsets $F_1 \subset \dots \subset F_{j-1}$ such that (4.14) holds for $k = 1, \dots, j-1$, we apply Lemma 4.7 with $G = F_{j-1} \cup F_j^*$: this yields a finite $H_j \subset \mathbb{S}$ such that $F_j := F_{j-1} \cup F_j^* \cup H_j$ is $\frac{1}{2}$ -balanced.

The inclusion $\text{supp } \alpha \subset \bigcup_{j \geq 1} F_j$ with such a sequence $\{F_j\}_{j \geq 1}$ implies $\alpha = 0$, by virtue of Lemma 4.6. \square

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